Few-Body Syst (2016) 57:1147–1153
DOI 10.1007/s00601-016-1151-8

Arkadiusz Kuroś · Przemysław Kościk · Jayanta K. Saha

Doubly Excited Resonance States of Helium Atom: Complex *Entropies*

Received: 13 April 2016 / Accepted: 23 August 2016 / Published online: 2 September 2016

Abstract We provide a diagonal form of a reduced density matrix of S-symmetry resonance states of two electron systems determined under the framework of the complex scaling method. We have employed the variational Hylleraas type wavefunction to estimate the complex *entropies* in doubly excited resonance states of helium atom. Our results are in good agreement with the corresponding ones determined under the framework of the stabilization method (Lin and Ho in Few-Body Syst 56:157, 2015).

1 Introduction

Last few years produced bulk of studies on quantum entanglement of few-body systems because of the development of new experimental techniques that have opened up perspectives for their applications in quantum information technology [1]. In particular, there has been a remarkable increase of interest in exploring properties of various quantum composite systems [2–17]. In recent years, both the entanglement and Shannon Entropy in helium and helium-like ions have also accelerated a great research attention [18–33]. However, studies on the correlation in resonance states have not yet drawn much attention. Pont et al. [34] proposed to associate the entanglement of a resonance state with a real part of a linear entropy computed from complex-valued eigenvalues of a reduced density matrix $\hat{\rho}$ determined in the framework of the complex scaling method (CSM) [35–38]. Within the framework of this approach, the properties of two-electron one-dimensional (1D) quantum dots near the autoionization thresholds have recently been investigated in [39]. The latest researches performed by Lin et. al. treated the quantum entanglement for doubly excited resonance states in helium and in the positronium negative ion under the framework of the stabilization method, [28] and [29], respectively.

A recent study [40], performed by one of us, has made a detailed discussion on the diagonal representation of the reduced density matrix for resonance states by using the CSM. Following to [35,41], the real and imaginary parts of an expectation value of a given complex-scaled operator give the average value of the quantity under consideration, and its uncertainty, respectively. In a strict mathematical sense, the value of the von Neumann (linear) entropy can be given as the expectation value of the operator $\log_2\hat{\rho}\,(\hat{1}-\hat{\rho})$. However, it must be stressed that the entropy cannot rigorously be treated as the expectation value of a quantal observable [42]. Therefore, in a resonance case, the interpretation of the real and imaginary parts of the expectation value of $\log_2\hat{\rho}\,(\hat{1}-\hat{\rho})$ as the entropy and its uncertainty, respectively, is problematic. Despite of this fact, we call the average values of the operators $\log_2\hat{\rho}$ and $\hat{1}-\hat{\rho}$ as the complex-valued von Neumann and linear *entropies*, respectively, and, following to [34,39,40], use them to characterize the correlation in resonance states. The

A. Kuroś (✉) · P. Kościk
Institute of Physics, Jan Kochanowski University, ul. Świętokrzyska 15, 25-406 Kielce, Poland
E-mail: akuros@ujk.edu.pl

J. K. Saha
Aliah University, II-A/27, Newtown, Kolkata 700156, India

purpose of this brief paper is to determine the complex *entropies* for doubly excited resonance states of helium, $2s^2\,{}^1S^e$ and $2p^2\,{}^1S^e$, and compare the results obtained with the corresponding ones determined in different ways [27,28].

This paper is organized as follows. In Sect. 2, we derive a diagonal representation of the complex reduced density matrix for the singlet S-symmetry resonance states. In Sect. 3, we determine the complex linear and von Neumann *entropies* for doubly excited resonance states $2s^2\,{}^1S^e$ and $2p^2\,{}^1S^e$ and some concluding remarks are left for Sect. 4.

2 Method

For the system under consideration here, the complex-scaled Hamiltonian is of the form [43]

$$\hat{H}^\theta = \sum_{j=1}^{2}\left[-\frac{e^{-2i\theta}}{2}\nabla_j^2 - \frac{Ze^{-i\theta}}{r_j}\right] + \frac{e^{-i\theta}}{r_{12}}, \tag{1}$$

where θ is the so-called scaling angle. A key point of the CSM is that the complex-scaled Hamiltonian is a non-Hermitian operator and the inner product is defined as

$$\ll\psi|\varphi\gg = \int_{\text{all space}} \psi(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}. \tag{2}$$

In the case of the resonance states, the density operator is defined as

$$\hat{\rho} = |\psi^R\gg\ll\psi^L|, \tag{3}$$

where $|\psi^R\gg$ and $\ll\psi^L|$ are right and left eigenstates of $\hat{H}^\theta$, which are determined by the following eigenequations $\hat{H}^\theta|\psi_i^R\gg = E_i(\theta)|\psi_i^R\gg$ and $\ll\psi_i^L|\hat{H}^\theta = \ll\psi_i^L|E_i(\theta)$, respectively, $\ll\psi_i^L|\psi_j^R\gg = \delta_{ij}$. The parameters of a given resonance level are determined at the angle $\theta = \theta_{opt}$ at which

$$\left.\frac{dE_i(\theta)}{d\theta}\right|_{\theta=\theta_{opt}} = 0.$$

The complex reduced density matrices for particles 1 and 2 are given by $\hat{\rho}_{12} = Tr_{21}[|\psi^R\gg\ll\psi^L|]$. An essential point is that the Hamiltonian (1) does not change upon permutation of the coordinates of particles, so that its right and left eigenfunctions can be expressed in an identical form $\ll\psi^L|\mathbf{r}\gg = \ll\mathbf{r}|\psi^R\gg = \psi$, $\ll\psi|\psi\gg = 1$ [40,44]. Here we restrict our study to the singlet S-symmetry states, the wavefunctions of which depend only on the radial coordinates r_1, r_2 and the inter-electronic angle coordinate θ_{12}, $\psi(\mathbf{r}_1,\mathbf{r}_2) = \psi(r_1,r_2,\cos\theta_{12})$. First we derive a diagonal form of the corresponding complex reduced density matrix with respect to the inner product (2). For the sake of brevity, we start with the Fourier-Legendre series representation of ψ, namely

$$\psi(r_1,r_2,\cos\theta_{12}) = \sum_{l=0}\frac{f_l(r_1,r_2)}{r_1r_2}P_l(\cos\theta_{12}), \tag{4}$$

where

$$\begin{aligned} f_l(r_1,r_2) &= r_1r_2\frac{2l+1}{2}\int_0^\pi \psi(r_1,r_2,\cos\theta_{12})P_l(\cos\theta_{12})\sin\theta_{12}d\theta_{12} \\ &= r_1r_2\frac{2l+1}{2}\int_{-1}^{1}\psi(r_1,r_2,t)P_l(t)dt. \end{aligned} \tag{5}$$

Further we proceed similarly as in the case of a two-boson wavefunction in [40]. Here we expand the function f_l into the basis of orthonormal Laguerre basis functions, namely

$$f_l(r_1,r_2) = \sum_{ij}a_{ij}^{(l)}L_i(r_1)L_j(r_2), \tag{6}$$

with

$$L_n(r) = \sqrt{\omega} e^{-\frac{r\omega}{2}} {}_1F_1(-n; 1; r\omega), \tag{7}$$

where the coefficient $a_{ij}^{(l)}$ are given by the following integrals

$$a_{ij}^{(l)} = \frac{2l+1}{2} \int_0^\infty \int_0^\infty \int_{-1}^1 r_1 r_2 \psi(r_1, r_2, t) P_l(t) L_i(r_1) L_j(r_2) dt dr_1 dr_2. \tag{8}$$

Since the function f_l is symmetric under the permutations of coordinates, the matrix $\mathbf{A}_l = [a_{ij}^{(l)}]$ is symmetric, which means that its eigenvector matrix $\mathbf{V}_l$ ($\mathbf{V}_l^{-1} = \mathbf{V}_l^T$) and the corresponding eigenvalues $\mathbf{D}_l = diag(d_{0l}, d_{1l}, \ldots,)$ satisfy $\mathbf{A}_l = \mathbf{V}_l \mathbf{D}_l \mathbf{V}_l^T$. After rewriting this formula as

$$a_{ij}^{(l)} = \sum_n (\mathbf{V}_l)_{in} d_{nl} (\mathbf{V}_l)_{jn}, \tag{9}$$

and substituting it into (6), we obtain

$$f_l(r_1, r_2) = \sum_n d_{nl} v_{nl}(r_1) v_{nl}(r_2), \tag{10}$$

where

$$v_{nl}(r) = \sum_k (\mathbf{V}_l)_{kn} L_k(r). \tag{11}$$

Hence, the substitution of Eq. (10) together with the well-known formula:

$$P_l(\cos\theta_{12}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{m=l} Y_{lm}(\varphi_1, \theta_1) Y_{lm}^*(\varphi_2, \theta_2),$$

into (4) gives

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \sum_{nl} \sum_{m=-l}^{m=l} k_{nl} u_{nlm}(\mathbf{r}_1) u_{nlm}^*(\mathbf{r}_2), \tag{12}$$

where $u_{nlm}(\mathbf{r}) = \frac{v_{nl}(r)}{r} Y_{lm}(\varphi, \theta)$ and $k_{nl} = \frac{4\pi d_{nl}}{2l+1}$, $\ll u_{nlm}^* | u_{n'l'm'} \gg = \delta_{nn'} \delta_{ll'} \delta_{mm'}$. As a result, the reduced density matrix for particles 1 or 2, takes a form

$$\hat{\rho}_{12} = \sum_{nl} \sum_{m=-l}^{m=l} \lambda_{nl} | u_{nlm} \gg \ll u_{nlm}^* |, \tag{13}$$

where $\lambda_{nl} = k_{nl}^2$. The vectors $\ll u_{nlm}^*|$ and $|u_{nlm} \gg$ are nothing else that the left and right eigenstates of $\hat{\rho}_{12}$, respectively, with the complex eigenvalues λ_{nl}, $\ll u_{nlm}^* | \hat{\rho}_{12} | u_{n'l'm'} \gg = \lambda_{nl} \delta_{nn'} \delta_{ll'} \delta_{mm'}$. Moreover, all of them, but the ones with $l = 0$ are $2l + 1$- fold degenerate so that the normalization condition gives $\sum_{nl} (2l + 1) \lambda_{nl} = 1$. From here on, we call λ_{nl} complex occupancies and in terms of them, the von Neumann and linear *entropies* have the forms [7]

$$S = -\sum_{nl} (2l+1) \lambda_{nl} \log_2 \lambda_{nl}, \tag{14}$$

and

$$L = 1 - \sum_{nl} (2l+1) \lambda_{nl}^2, \tag{15}$$

respectively. Following to [40], we associate the real and imaginary parts of the complex *entropy* as the amount of correlation and its uncertainty, respectively.

3 Results and Discussion

To compute the S-state energies, the widths and resonance eigenfuctions as well, we have used the Hylleraas basis functions

$$\psi^{Hyl} = \sum_{nmp} c_{nmp} e^{-\alpha s} s^n t^{2m} u^p, \tag{16}$$

with $0 \leq n+m+p \leq N$, where $s = r_1 + r_2$, $t = r_1 - r_2$, $u = r_{12} = (r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_{12})^{\frac{1}{2}}$, which leads to the eigenproblem:

$$[\mathbf{H}^{(\theta)} - E\mathbf{S}]\mathbf{C} = 0, \tag{17}$$

where $\mathbf{H}^{(\theta)} = [\langle n'm'p'|\hat{H}^{\theta}|nmp\rangle]$ and $\mathbf{S} = [\langle n'm'p'|nmp\rangle]$ are the Hamiltonian matrix and the overlap matrix, respectively. To determine the energies of resonance states $2s^2\,{}^1S^e$ and $2p^2\,{}^1S^e$ and their corresponding wavefunctions, we used an expansion (16) given by a 364-term ($N = 11$). Our results obtained at $\theta_{opt} = 0.3$, $\alpha = 1$: $E_{2s^2\,{}^1S^e} = -0.7778678 - 0.00227061i$, $E_{2p^2\,{}^1S^e} = -0.62192722 - 0.000107813i$ reproduce the corresponding literature values determined with a very high accuracy [45], that is, $E_{2s^2\,{}^1S^e} = -0.777867636 - 0.002270653i$, $E_{2p^2\,{}^1S^e} = -0.621927254 - 0.000107818i$, with to at least 7 digits. The calculation of the complex *entropies* requires determination of the coefficients d_{nl}. As described in Sect. 2, for a fixed value of l, the n_mth order approximations to the n_m lowest coefficients d_{nl} can be obtained by diagonalizing the matrix $\mathbf{A}_l = [a_{ij}^{(l)}]_{n_m \times n_m}$, the elements of which are given by the three-dimensional integrals (8), which we treat here numerically. Having the coefficients d_{nl} determined in that way for l up to l_m, we obtain thus the approximations to the true values of S and L as $S_{ap} = -\sum_{n,l}^{n_m-1,l_m}(2l+1)\lambda_{nl}\log_2\lambda_{nl}$ and $L_{ap} = 1 - \sum_{n,l}^{n_m-1,l_m}(2l+1)\lambda_{nl}^2$, respectively, where $\lambda_{nl} = (\frac{4\pi d_{nl}}{2l+1})^2$. The adjustable parameter ω appearing in (7) can be used to improve the convergence. A good criterion for convergence is the closeness of the sum of approximate complex occupancies to the theoretical value 1. We have verified by hand that a good rate of convergence is achieved at a parameter value $\omega = 2.5$ for both cases under consideration.

Table 1 Linear and von Neumann complex *entropies* for the state $2s^2\,{}^1S^e$ computed at $\omega = 2.5$

n_m	l_m	L_{ap}	S_{ap}	$\sum_{n,l}^{n_m-1,l_m}(2l+1)\lambda_{nl}$
10	0	$0.477362 + 0.00176401\,i$	$0.337396 - 0.00928007\,i$	$0.7227374 - 0.00207261\,i$
	1	$0.451993 + 0.00133491\,i$	$1.298414 - 0.00710255\,i$	$0.9995731 + 0.00002117\,i$
	2	$0.451993 + 0.00133491\,i$	$1.304087 - 0.00731422\,i$	$0.9999901 + 4.52956 * 10^{-6}i$
	3	$0.451993 + 0.00133491\,i$	$1.304263 - 0.00733733\,i$	$0.9999990 + 3.27358 * 10^{-6}i$
	4	$0.451993 + 0.00133491\,i$	$1.304294 - 0.00734341\,i$	$1.0000001 + 2.99857 * 10^{-6}i$
	5	$0.451993 + 0.00133491\,i$	$1.304303 - 0.00734536\,i$	$1.0000004 + 2.92017 * 10^{-6}i$
15	4	$0.451994 + 0.00133551\,i$	$1.304295 - 0.00735802\,i$	$0.9999999 + 5.33902 * 10^{-7}i$
	5	$0.451994 + 0.00133551\,i$	$1.304305 - 0.00735984\,i$	$1.0000003 + 4.58637 * 10^{-7}i$
20	4	$0.451994 + 0.00133543\,i$	$1.304293 - 0.00735533\,i$	$0.9999994 + 8.59354 * 10^{-7}i$
	5	$0.451994 + 0.00133543\,i$	$1.304303 - 0.00735716\,i$	$0.9999999 + 7.84988 * 10^{-7}i$

Table 2 Linear and von Neumann complex *entropies* for the state $2p^2\,{}^1S^e$ computed at $\omega = 2.5$

n_m	l_m	L_{ap}	S_{ap}	$\sum_{n,l}^{n_m-1,l_m}(2l+1)\lambda_{nl}$
10	0	$0.908817 - 0.00290117\,i$	$0.777321 - 0.00071258\,i$	$0.319393 + 0.005675030\,i$
	1	$0.777520 - 0.00071459\,i$	$2.192127 - 0.00318651\,i$	$0.967059 - 0.000226222\,i$
	2	$0.777321 - 0.00071258\,i$	$2.429632 - 0.00416282\,i$	$0.999129 - 0.000388452\,i$
	3	$0.777321 - 0.00071258\,i$	$2.443242 - 0.00421897\,i$	$1.000160 - 0.000392804\,i$
	4	$0.777321 - 0.00071258\,i$	$2.445791 - 0.00423267\,i$	$1.000320 - 0.000393569\,i$
	5	$0.777321 - 0.00071258\,i$	$2.446525 - 0.00423974\,i$	$1.000350 - 0.000393906\,i$
15	4	$0.777501 - 0.00088995\,i$	$2.445411 - 0.00375428\,i$	$0.999936 + 5.24362 * 10^{-7}i$
	5	$0.777501 - 0.00088995\,i$	$2.446149 - 0.00375853\,i$	$0.999975 + 3.16617 * 10^{-7}i$
20	4	$0.777500 - 0.00088971\,i$	$2.445416 - 0.00375074\,i$	$0.999938 + 1.26021 * 10^{-7}i$
	5	$0.777500 - 0.00088971\,i$	$2.446156 - 0.00375410\,i$	$0.999977 - 4.65872 * 10^{-8}i$

Table 3 Comparison of complex eigenenergy, linear and von Neumann complex *entropies* for $2s^2\,{}^1S^e$ and $2p^2\,{}^1S^e$ states of helium atom

	$Re[E]$	$Im[E]$	$Re[L_{ap}]$	$Re[S_{ap}]$
Ref. [27]				
$2s^2\,{}^1S^e$	−0.778775			0.882
$2p^2\,{}^1S^e$	−0.622559			1.346
Ref. [28]				
$2s^2\,{}^1S^e$	−0.7778583	−0.0022875	0.461704	1.378501
$2p^2\,{}^1S^e$	−0.6219259	−0.000105	0.777653	2.450665
Ref. [45]				
$2s^2\,{}^1S^e$	−0.777867636	−0.002270653		
$2p^2\,{}^1S^e$	−0.621927254	−0.000107818		
This work				
$2s^2\,{}^1S^e$	−0.7778678	−0.00227061	0.451994	1.304303
$2p^2\,{}^1S^e$	−0.62192722	−0.000107813	0.777500	2.446156

In Tables 1 and 2 our results obtained for L_{ap} and for S_{ap} together with the results for the sum of approximate occupancies are presented as functions of the cut-offs n_m and l_m, for the $2s^2\,{}^1S^e$ and $2p^2\,{}^1S^e$ states, respectively. Note that already at $n_m = 10$, fairly stable results were achieved at $l_m = 1$ in the former case and at $l_m = 2$ in the latter one. We can see that the real and imaginary parts of the sum of approximate occupancies tend to one and zero, respectively, as n_m and l_m are increased, which confirms the correctness of our calculations. It is evident from the tables that in the case of $2s^2\,{}^1S^e$, the terms higher than $l = 0$ in (4) contribute very little, which means that the angular correlation effects are expected to be very poor in this case. As expected, the situation is different in the case of $2p^2\,{}^1S^e$, where, in contrast, the most of electrons correlation comes mainly from the first angular level $l = 1$, which means that the angular correlations contribute importantly to the total correlation of the state. In Table 3 our results are compared with the ones obtained by two other methods [27,28], where, for the sake of completeness, we also added the reference values for the energy. We find that the real parts of the complex *entropies* coincide fairly well with the results obtained by C.H. Lin et al. by the stabilization method [28]. Both the results obtained by the Lin et al. and our results differ considerably from the ones obtained by Restrepo Cuartas et al. [27]. We suspect that the reason for this lies in the fact that probably the J.P. Restrepo Cuartas et al. have not taken into account in their calculations the degeneracies with respect to the magnetic quantum number. Note that the imaginary part of the complex linear *entropy* is positive for the $2s^2\,{}^1S^e$ state and negative for the $2p^2\,{}^1S^e$ state. To understand the physical meaning of the sign of the imaginary part of the complex *entropy*, more constructive analysis is needed, and this will be a subject of our further research.

4 Summary

In summary, we derived the diagonal representation of the complex reduced density matrix of the singlet S-symmetry states based on the Fourier-Legendre series. Subsequently, we have computed the complex *entropies* for doubly excited resonance states of helium atom. Our results for the real parts of the complex *entropies* are in a very good agreement with those determined under the framework of the stabilization method by C.H. Lin et al. [28]. The approach presented in this paper can be an alternative to the one based on the stabilization method when determining the real parts of the linear and von Neumann entropies of S-symmetry resonance states.

References

1. Nielsen, N., Chuang, I.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
2. Tichy, M.C., Mintert, F., Buchleitner, A.: Essential entanglement for atomic and molecular physics. J. Phys. B: At. Mol. Opt. Phys. **44**, 192001 (2011)
3. Boguslawski, K., Tecmer, P.: Orbital entanglement in quantum chemistry. Int. J. Quantum Chem. **115**, 1289 (2015)
4. Coe, J.P., Sudbery, A., D'Amico, I.: Entanglement and density-functional theory: testing approximations on Hooke's atom. Phys. Rev. B **77**, 205122 (2008)
5. Tennie, F., et al.: Pinning of fermionic occupation numbers: general concepts and one spatial dimension. Phys. Rev. A **93**, 042126 (2016)
6. Harshman, N.L.: One-dimensional traps, two-body interactions, few-body symmetries: I. One, two, and three particles. Few-Body Syst. **57**, 11 (2016)
7. Kościk, P.: Entanglement in S states of two-electron quantum dots with Coulomb impurities at the center. Phys. Lett. A **377**, 2393 (2013)
8. Kościk, P.: Quantum entanglement of two harmonically trapped dipolar particles. Few-Body Syst. **56**, 107 (2015)
9. Sowiński, T., Gajda, M., Rzążewski, K.: Pairing in a system of a few attractive fermions in a harmonic trap. EPL **109**, 26005 (2015)
10. García-March, M.A., Dehkharghani, A.S., Zinner, N.T.: Entanglement of an impurity in a few-body one-dimensional ideal Bose system. J. Phys. B: At. Mol. Opt. Phys. **49**, 075303 (2016)
11. Dehkharghani, A.S., Volosniev, A.G., Zinner, N.T.: Impenetrable mass-imbalanced particles in one-dimensional harmonic traps. J. Phys. B: At. Mol. Opt. Phys. **49**, 085301 (2016)
12. Nagy, I., Glasser, M.L.: Information-theoretic aspects of friction in the quantum mechanics of an interacting two-electron harmonic atom. J. Math. Chem. **53**, 1274 (2015)
13. Glasser, M.L., Nagy, I.: Exact evaluation of entropic quantities in a solvable two-particle model. Phys. Lett. A **377**, 2317 (2013)
14. Yaüez, R.J., Plastino, A.R., Dehesa, J.S.: Quantum entanglement in a soluble two-electron model atom. Eur. Phys. J. D **56**, 141 (2010)
15. Peng, H.T., Ho, Y.K.: Entanglement for excited states of ultracold bosonic atoms in one-dimensional harmonic traps with contact interaction. Mod. Phys. Lett. B **29**, 1550189 (2015)
16. Xu, F., et al.: Quantum tunneling effect in entanglement dynamics. Int. J. Quantum Chem. **116**, 7 (2016)
17. Pęcak, D., Sowiński, T., Gajda, M.: Two-flavor mixture of a few fermions of different mass in a one-dimensional harmonic trap. New J. Phys. **18**, 013030 (2016)
18. Dehesa, J., et al.: Quantum entanglement in helium. J. Phys. B: At. Mol. Opt. Phys. **45**, 015504 (2012)
19. Kościk, P., Okopińska, A.: Entanglement entropies in the ground states of helium-like atoms. Few-Body Syst. **55**, 1151 (2014)
20. López-Rosa, S., et al.: Quantum entanglement of helium-like systems with varying-Z: compact state-of-the-art CI wave functions. J. Phys. B: At. Mol. Opt. Phys. **48**, 175002 (2015)
21. Esquivel, R.O., López-Rosa, S., Dehesa, J.S.: Correlation energy as a measure of non-locality: quantum entanglement of helium-like systems. EPL **111**, 40009 (2015)
22. Lin, Y.C., Ho, Y.K.: Quantum entanglement for two electrons in the excited states of helium-like systems. Can. J. Phys. **93**, 646 (2015)
23. Lin, C.H., Ho, Y.K.: Quantification of entanglement entropy in helium by the SchmidtSlater decomposition method. Few-Body Syst. **55**, 1141 (2014)
24. Lin, Y.C., Lin, C.Y., Ho, Y.K.: Spatial entanglement in two-electron atomic systems. Phys. Rev. A **87**, 022316 (2013)
25. Lin, C.H., Lin, C.Y., Ho, Y.K.: Quantification of linear entropy for quantum entanglement in He, H- and Ps-ions using highly-correlated Hylleraas functions. Few-Body Syst. **54**, 2147 (2013)
26. Lin, C.H., Ho, Y.K.: Calculation of von Neumann entropy for hydrogen and positronium negative ions. Phys. Lett. A **378**, 2861 (2014)
27. Cuartas, J.P.R., Sanz-Vicario, J.L.: Information and entanglement measures applied to the analysis of complexity in doubly excited states of helium. Phys. Rev. A **91**, 052301 (2015)
28. Lin, C.H., Ho, Y.K.: Quantification of entanglement entropies for doubly excited states in helium. Few-Body Syst. **56**, 157 (2015)
29. Lin, C.H., Ho, Y.K.: Quantum entanglement and Shannon information entropy for the doubly excited resonance state in positronium negative ion. Atoms **3**, 422 (2015)
30. Hofer, T.S.: On the basis set convergence of electronelectron entanglement measures: helium-like systems. Front. Chem. **1**, 24 (2013)
31. Lin, Y.C., Fang, T.K., Ho, Y.K.: Quantum entanglement for helium atom in the Debye plasmas. Phys. Plasm. **22**, 032113 (2015)
32. Ho, Y.K., Lin, C.H.: Quantification of entanglement entropies for doubly excited resonance states in two-electron atomic systems. J. Phys. Conf. Ser. **635**, 092025 (2015)
33. Lin, C.H., Ho, Y.K.: Shannon information entropy in position space for two-electron atomic systems. Chem. Phys. Lett. **633**, 261 (2015)
34. Pont, F.M., Osenda, O., Toloza, J.H., Serra, P.: Entropy, fidelity, and double orthogonality for resonance states in two-electron quantum dots. Phys. Rev. A **81**, 042518 (2010)
35. Moiseyev, N.: Non-Hermitian Quantum Mechanics. Cambridge University Press, Cambridge (2011)
36. Reinhardt, W.P.: Complex coordinates in the theory of atomic and molecular structure and dynamics. Ann. Rev. Phys. Chem. **33**, 223 (1982)
37. Junker, B.R.: Recent computational developments in the use of complex scaling in resonance phenomena. Adv. Atom. Mol. Phys **18**, 208 (1982)

38. Ho, Y.K.: The method of complex coordinate rotation and its applications to atomic collision processes. Phys. Rep. **99**, 1 (1983)
39. Kuroś, A., Okopińska, A.: Entanglement properties of the two-electron quasi-one dimensional Gaussian quantum dot. Few-Body Syst. **56**, 853 (2015)
40. Kościk, P.: On the bipartite correlations in quantum resonance states. Phys. Lett. A **380**, 1256 (2016)
41. Berggren, T.: Expectation value of an operator in a resonant state. Phys. Lett. B **373**, 1 (1996)
42. Van der Merwe, A.: Old and new questions in physics, cosmology, philosophy, and theoretical biology. Plenum Press, New York (1983)
43. Ho, Y.K.: Complex-coordinate calculations for doubly excited states of two-electron atoms. Phys. Rev. A **23**, 2137 (1981)
44. Moiseyev, N.: Quantum theory of resonances: calculating energies, widths and cross-sections by complex scaling. Phys. Rep. **302**, 211 (1998)
45. Burgers, A., Wintgen, D., Rest, J.M.: Highly doubly excited S states of the helium atom. J. Phys. B: At. Mol. Opt. Phys. **28**, 3163 (1995)